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A numerical evaluation of the Finite Monkeys Theorem

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ABSTRACT

The Infinite Monkeys Theorem has long-established the eventual certainty of the complete works of William Shakespeare being reproduced by a monkey randomly pressing keys on a typewriter. This only considers the infinite limit, with either an infinite number of monkeys and/or an infinite time period of monkey labour. Here, we consider the Finite Monkeys Theorem and look at the probability of a given string being typed by one of a finite number of monkeys within a finite time allocation consistent with estimates for the lifespan of our universe. We also calculate the expected number of keystrokes until a target string would first be produced. Given the expected time until the heat death of the universe, we demonstrate that the widely-accepted conclusion from the Infinite Monkeys Theorem is, in fact, misleading in our finite universe. As such, this places the theorem in a class of probabilistic problems or paradoxes, including the St. Petersburg paradox, Zeno's dichotomy paradox and the Ross–Littlewood paradox wherein the infinite-resource conclusions directly contradict those obtained when considering limited resources, however sizeable.

1. Introduction

"Alas, poor ape, how thou sweat'st!" (William Shakespeare, *Henry IV Part 2,* Act 2, Scene 4)

Since its origins over a century ago, the Infinite Monkeys Theorem has been one of the most widely-known thought experiments. The exact source of the theorem is unclear. Most commonly, it is attributed to Émile Borel [1] or Thomas Henry Huxley [2], but some sources credit its conceptual origins back to Aristotle's *Metaphysics* [3]. In its most popular form, the Infinite Monkeys Theorem states that if you had an infinite number of monkeys and/or an infinitely long-time period, then a monkey pressing keys at random on a typewriter would eventually be certain to reproduce the works of William Shakespeare.

Mathematically, proof of the theorem is a simple consequence of the Borel–Cantelli lemma [4]. In plain language, when an event (e.g. a monkey typing a set string of characters) occurs in a given trial with finite non-zero probability, then the probability that the event never occurs tends towards zero as the number of independent trials tends towards infinity, however improbable the event may be on a single trial.

More recently, there have been attempts to test the theorem empirically, both experimentally and in popular culture. These approaches have included both computer simulation of the conceptual model [5] and studies with live primates and keyboards [6]. Additionally, in the long-running television show *The Simpsons*, the industrialist Charles Montgomery Burns attempted this with multiple monkeys chained to typewriters but gave up when the best one produced was the near-Dickensian "It was the best of times, it was the blurst of times". Although the Infinite Monkeys Theorem has been well studied, the Finite Monkeys Theorem is less well understood [7,8]. Here, we examine the situation when only a finite number of monkeys and only a finite time period are available. We then calculate the probabilities of particular phrases being typed by a given number of monkeys in a given time period.

2. Materials and methods

Initially, we need to state the assumption of the model. We assume that a keyboard contains K different keys. Each typing monkey presses N keys (one at a time) such that each key is selected with equal probability on each press, independent of all other keys selected. There is a target string of text of length L characters which we are seeking to be typed. We first consider the mono-monkey combinatorics, i.e. just a single monkey typing at a time.

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3. Calculation

With N keystrokes, each of which is one of K different characters, there are K^N different ordered sequences which could be typed. We now seek how many of these (equally probable) sequences contain the L consecutive characters from the target string at least once.

Regarding the string of length L as one "character" we have to place this one "character" in one place in the string of $N - L + 1$ characters. The remaining $N - L$ characters can be any of the K possible characters. We therefore have $(N - L + 1)K^{N-L}$ ways for this to happen.

This, however, overcounts since it is possible that the target string of length L appears more than once in the N keystrokes. For example, if we are seeking the string *HH* in a string of five coin flips (i.e. *H* or *T*), the above method would count the outcome *HHTHH* twice, so we need to correct for this double counting.

To consider the extent of overcounting, we first regard the string of length L as one "character". We have to place this one "character" twice in the string of $N - 2L + 2$ characters. The remaining $N - 2L$characters can be any of the K possible characters.

We therefore have $\binom{N-2L+2}{2}K^{N-2L}$ ways for this to happen, where $\binom{N-2L+2}{2} = \frac{(N-2L+2)!}{2!(N-2L)!}$ denotes the binomial coefficient.

Subtracting these, however, fails to count any sequence in which the target string appears three or more times. Applying the inclusion-exclusion principle [9], we obtain the probability that the sequence of N keystrokes contains the target sequence of length L at least once is

$$\frac{1}{K^N}\left[\sum_{j=1}^{\left\lfloor \frac{N}{L} \right\rfloor}(-1)^{j+1}\binom{N-jL+j}{j}K^{N-jL}\right] \tag{1}$$

Here $\left\lfloor \frac{N}{L} \right\rfloor$ denotes the floor of $\frac{N}{L}$ i.e. the largest integer less than or equal to $\frac{N}{L}$, since this is the maximum number of times the string of length L could appear. This also assumes that multiple occurrences do not overlap i.e. the final word(s) of the first time the target string appears are not also the first word(s) of the second time the target string appears. In reality, with plain text and proper punctuation, this will almost certainly be the case.

The result in eq (1) gives the exact probability of a target sequence of length L being typed within N keystrokes. It is not, however, computationally tractable for many practical purposes, such as when L is very large. In such cases, we can derive an approximation for when $1 \ll N \ll K^L$

Let $P(N, L)$ be the probability that a sequence of N keystrokes does not contain the target sequence of length L. For large values of L and K we therefore have that $\frac{\partial P(N,L)}{\partial N} \approx -\frac{1}{K^L} + O\left(\frac{1}{K^{2L}}\right)$. This is effectively equivalent to truncating the series in eq (1) at its leading term.

From this, we can obtain a Taylor series [10] expansion $P(N + \Delta N, L) \approx P(N, L) - \Delta N \frac{1}{K^L}$.

We can therefore see that, approximating N to be a continuous variable, we have that the number of keystrokes until the first occurrence of the target string of length L scales according as an exponentially distributed [10] random variable with rate parameter $\frac{1}{K^L}$.

From this, we can easily extend the result to the multi-monkey combinatorics problem. With $M > 1$ monkeys working independently of one another, simultaneously typing in the same manner as the single monkey in the earlier problem, we have that the minimum of a set of independent exponential random variables is itself an exponential random variable with rate parameter equal to the sum of all rate parameters [11].

This can also be seen by noting that, for an exponential variable with rate parameter $\frac{1}{K^L}$ the probability of a single monkey having failed to

Table 1
The expected number of keystrokes needed until various phrases would be typed. and the probability of the sequence ever being before either the typist's death, the death of all currently living chimpanzees (if typing simultaneously) or the heat death of the universe. Where only a word count was available, we assumed an average of 5.7 characters per word, based on 4.7 word length in English [17] plus a single space or punctuation point.

Target Phrase	Expected number of keystrokes needed	Probability of occurrence within the lifetime of a single chimpanzee	Probability of occurrence before death of all chimpanzees	Probability of occurrence before universe heat death
"Bananas"	$30^7 \approx 2.2 \times 10^{10}$	$\approx \frac{10^9}{2.2 \times 10^{10}} \approx 0.05$	$1 - 0.95^{200000} \approx 1$	$1 - 0.95^{6.4\times10^{103}} \approx 1$
"I chimp, therefore I am"	$30^{23} \approx 9.4 \times 10^{33}$	$\approx 10^{-25}$	$\approx 2 \times 10^{-20}$	≈ 1
Entire text of *Curious George* by Margret Rey and H. A. Rey (around 1800 words)	$\approx 30^{(1800\times5.7)}$ $\approx 10^{15155}$	$\approx 10^{-15146}$	$\approx 2 \times 10^{-15141}$	$\approx 6.4 \times 10^{-15043}$
Entire text of *Planet of the Apes* by Pierre Boulle (around 83,000 words)	$\approx 30^{(83000\times5.7)}$ $\approx 10^{698826}$	$\approx 10^{-698817}$	$\approx 2 \times 10^{-698812}$	$\approx 6.4 \times 10^{-698714}$
Entire works of William Shakespeare (around 884,647 words) [16]	$\approx 30^{(884647\times5.7)}$ $\approx 10^{7448366}$	$\approx 10^{-7448357}$	$\approx 2 \times 10^{-7448352}$	$\approx 6.4 \times 10^{-7448254}$

produce the target string before time N is approximately $e^{-\frac{N}{K^L}}$ hence the probability that M independently-working moneys have not produced the string is the product of M such probabilities, so is $\left(e^{-\frac{N}{K^L}}\right)\left(e^{-\frac{N}{K^L}}\right)\ldots\left(e^{-\frac{N}{K^L}}\right) = e^{-\frac{MN}{K^L}}$. From this, we know that the time until M $(1 < M \ll K^L)$ monkeys first type the target string scales as an exponential random variable with rate parameter $\frac{M}{K^L}$, so the expected number of keys pressed before the string is generated is approximately $\frac{K^L}{M}$.

4. Results

In order to quantify the probabilities or expected timescales, we need to make a few additional assumptions. We assume that the keyboard contains $K = 30$ different keys, covering all letters of the English language, plus some common punctuation. For the timescale calculations, we assume that a monkey typist presses one key per second every second of the day. As chimpanzees are human's closest relative amongst apes [12], we focus on these as the species of study. We take a chimpanzee's working lifespan to be 10^9 s (just over 30 years [13]) and the heat death of the universe to occur 10^{100} years [14] after the experiment begins. Assuming that the current population of around 200,000 chimpanzees [15] remains constant until the end of the universe, we therefore have a maximum of around $10^{100} \times 3.2 \times 10^7 \times 2 \times 10^5/(10^9) = 6.4 \times 10^{103}$ working chimpanzee lives, since there are approximately 3.2×10^7seconds in a year. There is no consideration to breeding or the quantity of food required for the population to persist without decline.

Table 1 quantifies the expected number of keystrokes until a randomly-typing monkey first produces a given target string, for several different strings, across ranges of complexity from the single word "Bananas" to the complete works of William Shakespeare. These are visualised in Fig. 1 on a log(log) scale, as is required for values spanning such hugely different orders of magnitude. Probabilities are numerically evaluated for three timescales: within the lifespan of a single chimpanzee, within the lifespan of all chimpanzees where no breeding occurs to prolong the population and within the heat death of the universe, assuming a persisting monkey population until that time.

From this, we can see that all but the most trivial of phrases will, in fact, almost certainly never be produced during the lifespan of our universe. There are many orders of magnitude difference between the expected numbers of keys to be randomly pressed before Shakespeare's works are reproduced and the number of keystrokes until the universe collapses into thermodynamic equilibrium (Fig. 2). It is not plausible that, even with possible improved typing speeds or an increase in chimpanzee populations, these orders of magnitude can be spanned to the point that monkey labour will ever be a viable tool for developing written works of anything beyond the trivial. As such, we reject the conclusions from the Infinite Monkeys Theorem as potentially misleading within our finite universe. This evaluation effectively contradicts one of the best-known pieces of folk mathematics but also firmly places the Infinite Monkeys Theorem alongside other seeming paradoxes [18-21] with contradictory results in the finite and infinite cases.

5. Discussion

This is not the first time that monkey labour has been considered as a potential source of intelligible output. In 1964, the science fiction writer and futurist Arthur C Clarke foresaw that "with our present knowledge of animal psychology, and genetics, we could certainly solve the servant problem with the help of the monkey kingdom. Of course, eventually our super-chimpanzees would start forming trade unions, and we'd be right back where we started." [22] Despite this fear, to the best of our

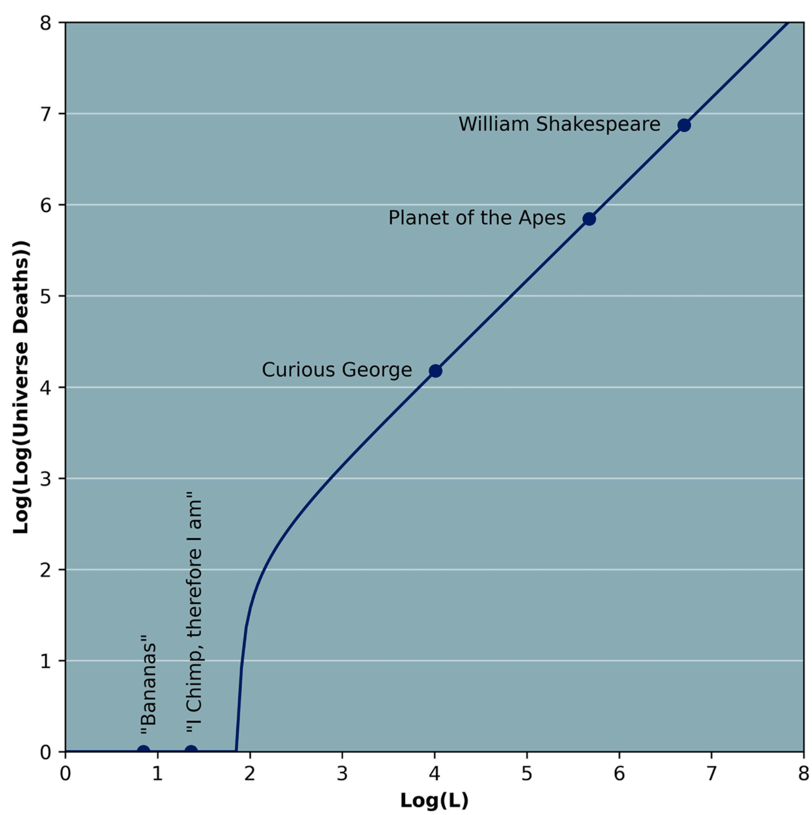

Fig. 1. The expected number of keystrokes needed until each of the phrases presented in Table 1 would first be produced. Note the vertical scale, which is in units of $\log_{10}(\log_{10})$ universe deaths, truncated at 0. Points above 1 on this axis correspond to strings which are almost certain never to be typed before the universe ends.

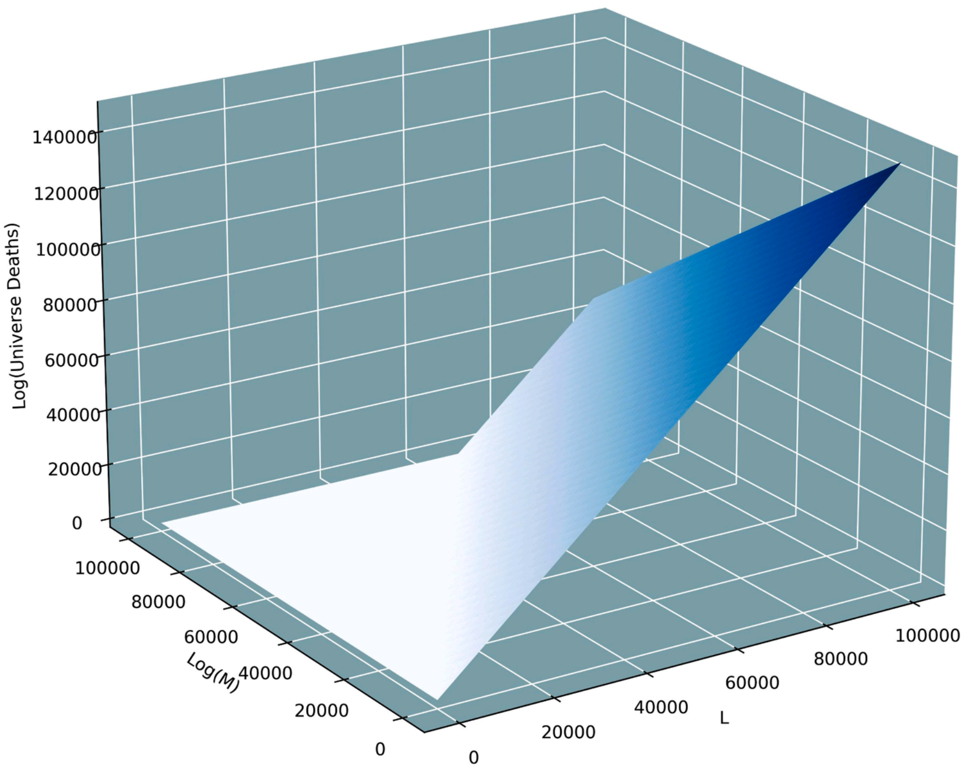

Fig. 2. The expected number of keystrokes required to produce target strings of lengths up to L of 100,000. Note that the current chimpanzee population of around 200,000 is equivalent to $\log_{10}(M)$ of around 5.3.

knowledge, no chimpanzee has ever taken industrial action to date, regardless of pay or workplace conditions.

It should also be noted here that we are not considering the philosophical question of whether two identical strings of text, one intentionally generated by a cognisant creator and the other generated unintentionally should be considered as the same output [23]. This very same abstract problem is currently very pertinent, albeit more in the context of Generative Artificial Intelligence and less in that of chimpanzees. Even with this assumption, we clearly demonstrate that the conclusions from the Finite Monkeys Theorem are at odds with those of the more famous Infinite Monkeys Theorem, which are not applicable in our finite universe.

6. Conclusion

Given plausible estimates of the lifespan of the universe and the amount of possible monkey typists available, this still leaves huge orders of magnitude differences between the resources available and those required for non-trivial text generation. As such, we have to conclude that Shakespeare himself inadvertently provided the answer as to whether monkey labour could meaningfully be a replacement for human endeavour as a source of scholarship or creativity.

To quote *Hamlet*, Act 3, Scene 3, Line 87: “No”.

CRediT authorship contribution statement

Stephen Woodcock: Writing – review & editing, Writing – original draft, Validation, Supervision, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Jay Falletta:** Writing – review & editing, Visualization, Validation, Software, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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